

Queen Mary, University of London
BSc Examination by Course Units

MAS101 Calculus I

Tuesday, 10 May 2005, 10 am

Duration: TWO HOURS

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators are NOT permitted in this examination. The unauthorised use of a calculator constitutes an examination offense.

SECTION A

Each question carries 8 marks. You should attempt ALL questions.

- A1. The functions $f(x)$ and $g(x)$ are defined for all real numbers x by:

$$f(x) = \cos x, \quad g(x) = x - 2x^3.$$

Calculate:

$$(i) \quad \frac{d}{dx}(f(x) \cdot g(x))$$

$$(ii) \quad \frac{d}{dx}f(g(x))$$

- A2. Evaluate the integral

$$\int e^x \cos(2x) dx$$

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A3. Compute the following limits

$$(i) \quad \lim_{x \rightarrow \infty} \frac{x^{12} - 3x^7 + 5}{9 + 4x^3 - 7x^{12}}$$

$$(ii) \quad \lim_{x \rightarrow 0} \frac{\sinh(5x) - 5x}{x^3}.$$

A4. Evaluate

$$\frac{d}{dx} \int_3^{\ln x} [5t^4 - 3e^t - 7] dt,$$

without attempting to perform the integral.

A5. Test the following series for convergence or divergence. Give reasons for your answers.

$$(i) \quad \sum_{n=1}^{\infty} \frac{5}{n+4}$$

$$(ii) \quad \sum_{n=1}^{\infty} \left(\frac{n+1}{n+2} \right) \left(\frac{3}{5} \right)^n$$

Section B questions continue overleaf

SECTION B

Each question carries 20 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best 3 questions will be counted.

- B1. (i) Derive the reduction formula for the integral

$$I_n = \int_0^{\pi/2} \cos^n x dx$$

where n is an integer greater than one, in the form

$$I_n = \left(\frac{n-1}{n} \right) I_{n-2}$$

- (ii) Hence show that when n is even

$$\int_0^{\pi/2} \cos^n x dx = \frac{(n-1)(n-3)\dots 3 \cdot 1}{n(n-2)\dots 4 \cdot 2} I_0.$$

Calculate I_0 .

- (iii) What happens when n is odd?

- B2. Plot the function

$$y = \frac{x^2}{x^2 + 1}.$$

Show any asymptotes, maxima, minima and inflection points that exist.

Section B questions continue overleaf

- B3. (i) State de Moivre's Theorem for both positive and negative integers.
- (ii) Use this theorem to express $\cos 3\theta$ and $\sin 3\theta$ in terms of powers of $\sin \theta$ and $\cos \theta$.
- (iii) Express $-\sqrt{3} + i$ as a complex number in the form $r(\cos \theta + i \sin \theta)$. Hence find the 5 complex roots of the complex equation

$$z^5 = -\sqrt{3} + i.$$

[It is sufficient to leave the solutions to this part in exponential form.]

- B4. (i) Write down the first three non-zero terms of the Maclaurin series for the function

$$f(x) = e^{-x^2}$$

- (ii) Write down the general term of the series and hence find the radius of convergence for this power series.

- B5. (i) State the mean value theorem and the intermediate value theorem for a function $f(x)$ defined on an interval $a \leq x \leq b$.
- (ii) By applying the mean value theorem to the function $\sin^2 x$ or otherwise, show that

$$|\sin^2 b - \sin^2 a| \leq |b - a|$$

for any real numbers a and b .

You may assume that $|\sin x| \leq 1$ for all x .

End of examination paper

Solutions

All are new examples of methods covered in class and homework.

A1. (i)

$$\frac{d}{dx}(f(x).g(x)) = -(x - 2x^3) \sin x + (1 - 6x^2) \cos x$$

(ii)

$$\frac{d}{dx}f(g(x)) = (1 - 6x^2) \sin(x - 2x^3)$$

A2. Can proceed in 2 ways:

Writing it as

$$\operatorname{Re} \left[\int e^{x(1+2i)} dx \right]$$

Or by integration by parts:

$$I = \int e^x \cos 2x dx = e^x \cos 2x - \int e^x (-2 \sin x) dx$$

and integrating by parts again ...

the result is

$$\frac{1}{5} [e^x \cos 2x + 2e^x \sin 2x] + D$$

A3. (i) divide top and bottom by x^{12} to give

$$-\frac{1}{7}$$

(ii) Use l'Hôpital rule repeatedly gives

$$\frac{125}{6}$$

A4. Let $u = \ln x$. Then using the fundamental theorem of calculus gives

$$= \frac{1}{x} [5(\ln x)^4 - 3x - 7]$$

A5. (i) Compare with the harmonic series

(ii) Using the Ratio Test eventually gives

$$\frac{3}{5} < 1$$

which shows that the series is convergent.

B1. Write integral as:

$$I_n = \int_0^{\pi/2} \cos x \cos^{n-1} x dx$$

and use integration by parts: see Notes.

B2. (i) The function is even.

The zero is at $(0, 0)$

For extrema look at

$$\frac{dy}{dx} = 0$$

which gives $x = 0$.

[2] To determine nature, look at the sign of $\frac{dy}{dx}$ around the point $x = 0$. This shows that $x = 0$ is a minimum point.

Alternatively, look at

$$\frac{d^2y}{dx^2}$$

which is positive at $x = 0$, again showing that $x = 0$ is a minimum point.

For possible points of inflection look at

$$\frac{d^2y}{dx^2} = 0$$

which gives $x = \pm 1/\sqrt{3}$. To check if these are points of inflection, look at the sign of $\frac{d^2y}{dx^2}$ around the points $x = \pm 1/\sqrt{3}$. As can be seen the signs change, so the points are points of inflection.

Finally for asymptotes, note

$$\lim_{x \rightarrow \pm\infty} \frac{x^2}{x^2 + 1} = 1$$

which gives the horizontal asymptote $y = 1$.

Putting all this information together obtain the graph of the function.

B3. (i) See notes.

(ii) Recall (using de Moivre's theorem)

$$(\cos 3\theta + i \sin 3\theta) = (\cos \theta + i \sin \theta)^3$$

(iii) Write RHS as a complex number in polar form, by calculating the modulus $r = \sqrt{3 + 1} = 2$ and the argument

$$\theta = \arctan \left[-\frac{1}{\sqrt{3}} \right] = \frac{5\pi}{6}$$

Hence

$$-\sqrt{3} + i = 2e^{5\pi i/6}$$

And then proceed as in notes and CW.

B4. (i) Maclaurin series for a function $f(x)$ is:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^n(0) + E_n(x)$$

where

$$E_n(x) = \frac{x^{n+1}}{(n+1)!} f^{n+1}(\theta x), \quad 0 < \theta < 1$$

For the function $f(x) = e^{-x^2}$, we have

$$\begin{aligned} f'(x) &= -2xe^{-x^2} \\ f''(x) &= -2e^{-x^2} + 4x^2e^{-x^2} \end{aligned}$$

and similarly higher derivatives that you must find.

These give which gives

$$f(0) = 1, \quad f'(0) = 0, \dots$$

Substituting in above, we obtain the first 3 non-zero terms of the series as:

$$e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} + \dots$$

(ii) The general term of the series is

$$(-1)^n \frac{(x)^{2n}}{n!}$$

For the radius of convergence look at

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right|$$

which is zero for all x . Hence the series converges for all x and the radius of convergence is ∞ .

B5. (i) The Intermediate Value Theorem states that if f is continuous on the interval $[a, b]$ and either $f(a) < 0 < f(b)$ or $f(b) < 0 < f(a)$, then there is c such that $a < c < b$ and $f(c) = 0$.

The Mean Value Theorem states that if f is continuous on $[a, b]$ and differentiable on (a, b) then there is c with $a < c < b$, such that

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

For solution see CWs.